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LIMITING LAWS FOR LONG BROWNIAN BRIDGES PERTURBED BY THEIR ONE-SIDED MAXIMUM, III

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In homage to Professors E. Csaki and P. Revesz.

Abstract. Results of penalization of a one-dimensional Brownian motion (X_t), by its one-sided maximum $(S_t = \sup_{0\le u\le t} X_u)$, which were recently obtained by the authors are improved with the consideration-in the present paper- of the asymptotic behaviour of the likewise penalized Brownian bridges of length t, as $t\to\infty$, or penalizations by functions of (S_t, X_t), and also the study of the speed of convergence, as $t\to\infty$, of the penalized distributions at time t.

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1 Introduction

1.1 Let $\big(\Omega = \mathcal{C}(\mathbb{R}_+,\mathbb{R}),\ (X_t)_{t\ge 0},\ (\mathcal{F}_t)_{t\ge 0}\big)$ be the canonical space with (X_t) the process of coordinates : $X_t(\omega)=\omega(t); t\ge 0$, $(\mathcal{F}_t)_{t\ge 0}$ the canonical filtration associated with (X_t). We write $\mathcal{F}_\infty$ for the σ-algebra generated by $\bigcup_{t\ge 0}\mathcal{F}_t$. Let P_0 be the Wiener measure defined on the canonical space such that $P_0(X_0=0)=1$.
In this paper, as well as in the previous ones ([14], [16], [15]), we consider perturbations of Brownian motion with certain processes $(F_t)_{t\ge 0}$, which we call weight-processes; precisely, let $(F_t)_{t\ge 0}$ be an

$(\mathcal{F}_t)$-adapted, non negative process, such that $0 < E_0(F_t) < \infty$, for any $t \geq 0$, and $Q^F_{0,t}$ the probability measure (p.m.) defined on $(\Omega,\ \mathcal{F}_t)$ as follows :

$$Q^F_{0,t}(\Gamma_t) := \frac{1}{E_0[F_t]}\ E_0[1_{\Gamma_t} F_t], \quad \Gamma_t \in \mathcal{F}_t. \tag{1.1}$$

We can interpret the p.m. $Q^F_{0,t}$ as the Wiener measure penalized by the weight F_t. We say that a penalization principle holds if there exists a p.m. Q^F_0 on $\big(\Omega, \mathcal{F}_\infty\big)$ such that $Q^F_{0,t}$ converges weakly to Q^F_0, as $t \to \infty$:

$$\lim_{t\to\infty} Q^F_{0,t}(\Gamma_u) = Q^F_0(\Gamma_u), \quad \text{for any } \Gamma_u \in \mathcal{F}_u,\ u \geq 0. \tag{1.2}$$

Throughout the paper, (S_t) stands for the one-sided maximum of (X_t) : $S_t := \max\limits_{0\leq u\leq t} X_u,\ t \geq 0.$
In fact, in our study, the following situation always occurs : let

$$M^{(t)}_u := \frac{1}{E_0[F_t]} E_0[F_t|\mathcal{F}_u], \quad u < t.$$

Then, we show that, for fixed u, $M^{(t)}_u$ converges a.s., with respect to P_0, to a variable M_u, such that $E_0[M_u] = 1$. Thus by Scheffé's lemma (see, e.g. [6], Chap. V, T21) $M^{(t)}_u$ converges in $L^1(P_0)$ towards M_u, which explains why (1.2) holds without any restriction on $\Gamma_u \in \mathcal{F}_u$.
1.2 In a series of papers ([14], [16], [15] and [10]) we have considered some classes of examples involving respectively for our weight-process (F_t) a function of :

- $\displaystyle\int_0^t V(X_s)ds$ where $V : \mathbb{R} \mapsto \mathbb{R}_+$.
- the unilateral maximum S_t; we have also treated the two-dimensional process (S_t, t).
- $(L^0_t; t \geq 0)$ the local time at 0 of $(X_t)_{t\geq 0}$.
- The triple $((S_t, I_t, L^0_t); t \geq 0)$, where (I_t) denotes the one-sided minimum : $I_t = -\min\limits_{0\leq u\leq t} X_u$.
- $(D_t; t \geq 0)$ the number of down-crossings of X from level b to level a.

In this paper we only consider the case : $F_t = f(X_t, S_t)$, where $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$. In particular if $F_t = \varphi(S_t)$, where $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defines a probability density, i.e.

$$\int_0^\infty \varphi(y)dy = 1, \tag{1.3}$$

our starting point is the following main result in [15] :

Theorem 1.1 *Let $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (1.3) and $\Phi(y) = \displaystyle\int_0^y \varphi(x)dx, y \geq 0$.*

1. *For every $u \geq 0$, and Γ_u in $\mathcal{F}_u$, the quantity :*

$$Q^\varphi_0(\Gamma_u) := \lim_{t\to\infty} \frac{E_0\big[1_{\Gamma_u}\varphi(S_t)\big]}{E_0\big[\varphi(S_t)\big]}, \tag{1.4}$$

exists; hence, Q^φ_0 may be extended as a p.m. on $\big(\Omega, \mathcal{F}_\infty\big)$.

2. *It is equal to $E_0[1_{\Gamma_u} M^\varphi_u]$, where $(M^\varphi_u)_{u\geq 0}$ is the martingale :*

$$M^\varphi_u = \varphi(S_u)(S_u - X_u) + 1 - \Phi(S_u); \quad u \geq 0. \tag{1.5}$$

(These $\big(P_0, (\mathcal{F}_u)\big)$-martingales have been introduced in [1]).

3. *The probability Q_0^φ may be disintegrated as follows :*

   (a) *under Q_0^φ, S_∞ is finite a.s., and admits φ as a probability density;*

   (b) *$Q_0^\varphi(S_\infty \in dy)$ a.e., conditionally on $S_\infty = y$, the law of (X_t), under Q_0^φ is equal to $Q_0^{(y)}$, where, for any $y > 0$, the p.m. $Q_0^{(y)}$ on the canonical space is defined as follows :*

      i. *$(X_t; t \leq T_y)$ is a Brownian motion started at 0, and considered up to T_y, its first hitting time of y,*

      ii. *the process $(X_{T_y+t}; t \geq 0)$ is a "three dimensional Bessel process below y", namely : $(y - X_{T_y+t}\ ;\ t \geq 0)$ is a three dimensional Bessel process started at 0.*

      iii. *the processes $(X_t; t \leq T_y)$ and $(X_{T_y+t}; t \geq 0)$ are independent.*

   (c) *Consequently :*

$$Q_0^\varphi(\Gamma | S_\infty = y) := Q_0^{(y)}(\Gamma), \qquad \text{for any } \Gamma \in \mathcal{F}_\infty, \tag{1.6}$$

$$Q_0^\varphi(\cdot) = \int_0^\infty Q_0^{(y)}(\cdot)\ \varphi(y)\, dy. \tag{1.7}$$

In the present paper, we develop a number of variants of this Theorem 1.1, by presenting either extensions or some new proofs of this theorem. Here are these variants, together with the organization of our paper.
In Section 2, we give, in particular, another proof of Theorem 1.1, which originates from the following considerations : the main step in [15] consisted in studying the asymptotics of $E[\varphi(S_t)|\mathcal{F}_s]$, for fixed s, as $t \to \infty$. In Section 2 here, we proceed in a dual manner by studying the asymptotics of

$$Q_{0,t}^{(y)}(\Gamma_u) := P(\Gamma_u | S_t = y), \tag{1.8}$$

as $t \to \infty$, where $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$ are fixed.

Theorem 1.2 *Let $y > 0$, $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$.*

1. *As $t \to \infty$, $Q_{0,t}^{(y)}(\Gamma_u)$ converges towards the probability $Q_0^{(y)}(\Gamma_u)$, where $Q_0^{(y)}$ is the probability introduced in Theorem 1.1, 3.*

2. *Moreover, $Q_0^{(y)}$ satisfies :*

$$Q_0^{(y)}(\Gamma_u) = e^{-y^2/2u}\sqrt{\frac{2}{\pi u}} E_0\Big[1_{\Gamma_u}(y - X_u)\Big| S_u = y\Big] + E_0[1_{\Gamma_u} 1_{\{S_u < y\}}]. \tag{1.9}$$

In Section 3, we strengthen the result obtained in Section 2, in that we consider the existence of the limits, as $t \to \infty$, of :

$$\frac{E_0\left[1_{\Gamma_u}\varphi(S_t)|X_t = a\right]}{E_0\left[\varphi(S_t)\big|X_t = a\right]} \tag{1.10}$$

and, in the spirit of the preceding Section 2 (or Theorem 1.2) :

$$Q_{0,t}^{a,y}(\Gamma_u) := P_0(\Gamma_u | X_t = a, S_t = y), \tag{1.11}$$

where $u \geq 0, \Gamma_u \in \mathcal{F}_u, y \geq a_+$.
The title of the present paper originates from this central Section 3. The results are the following :

- concerning (1.11), we obtain :

Theorem 1.3 *1. For any $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$,*

$$\lim_{t\to\infty} Q_{0,t}^{a,y}(\Gamma_u) := Q_0^{a,y}(\Gamma_u), \tag{1.12}$$

exists.

2. The p.m. $Q_0^{a,y}$ may be expressed as a convex combination of the laws $Q_0^{(z)}, z \in \mathbb{R}_+$:

$$(2y-a)Q_0^{a,y}(\cdot) = (y-a)Q_0^{(y)}(\cdot) + \int_0^y dz Q_0^{(z)}(\cdot). \tag{1.13}$$

Remark 1.4 *1. Recall that $Q_0^{(y)}(S_\infty = y) = 1$. Since $Q_0^{a,y}$ satisfies (1.13), we deduce :*

$$Q_0^{a,y}(S_\infty = y) = \frac{y-a}{2y-a}.$$

2. As we started with the Brownian bridge, we might have expected that, under the limiting p.m., some constraint involving the position of the process at infinity would hold. This is not the case, indeed, the parameter a only appears in the coefficients of the convex combination in (1.13) and $Q_0^{(y)}\big(\lim_{t\to\infty} X_t = -\infty\big) = 1$.

3. Identity (1.13) implies that $(a,y) \mapsto Q_0^{a,y}$ is continuous.

4. We may recover $Q_0^{(y)}$ from $\big(Q_0^{a,y}; y_+ \leq a\big)$ since $Q_0^{(y)} = \dfrac{d}{dy}\big(yQ_0^{y,y}\big)$.

5. Let $\mu^{a,y}$ the p.m. on $\mathbb{R}_+$: $\mu^{a,y}(dz) = \dfrac{y-a}{2y-a}\delta_y(dz) + \dfrac{1}{2y-a}1_{[0,y]}(z)dz$. The relation (1.13) admits the following probabilistic interpretation : first, z is chosen at random following $\mu^{a,y}$; secondly, the dynamics of (X_t) is given by $Q_0^{(z)}$.

6. From Lévy's theorem, under P_0, $\big((S_t - X_t, S_t; t \geq 0\big)$ and $\big((|X_t|, L_t^0; t \geq 0\big)$ have the same distribution. Let $\mathbb{Q}_0^{(y)}$ be the unique p.m. on $\big(\Omega, \sigma(|X_t|, t \geq 0)\big)$ satisfying :

$$\mathbb{Q}_0^{(y)}(\Gamma_u) = e^{-y^2/2u}\sqrt{\frac{2}{\pi u}}E_0\Big[1_{\Gamma_u}|X_u| \,\Big|L_u^0 = y\Big] + E_0[1_{\Gamma_u}1_{\{L_u^0 < y\}}], \tag{1.14}$$

for any $u \geq 0$ and $\Gamma_u \in \sigma(|X_t|, t \leq u)$.
In a forthcoming paper [13] it is proved that the analog of (1.12) and (1.13) is :

$$\lim_{t\to\infty} P_0(\Gamma_u \Big| \, |X_t| = a, L_t^0 = y) = \frac{a}{a+y}\mathbb{Q}_0^{(y)}(\Gamma_u) + \frac{1}{a+y}\int_0^y \mathbb{Q}_0^{(z)}(\Gamma_u)dz, \tag{1.15}$$

with Γ_u any event in $\sigma(|X_t|, t \leq u)$, and an adequate extension of this result with $|X_s|$ being replaced by a Bessel process with dimension $d < 2$ is obtained.

- As for (1.10), we obtain :

Theorem 1.5 *Let $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that :*

$$\int_0^\infty (1+x)\varphi(x)dx < \infty. \tag{1.16}$$

1. For any $u \geq 0$, $\Gamma_u \in \mathcal{F}_u$ and $a \in \mathbb{R}$, we have :

$$Q_0^{a,\varphi}(\Gamma_u) := \lim_{t\to\infty} \frac{E_0\big[1_{\Gamma_u}\varphi(S_t)|X_t = a\big]}{E_0\big[\varphi(S_t)|X_t = a\big]}, \tag{1.17}$$

exists.

2. *The p.m. $Q_0^{a,\varphi}$ may be expressed in terms of either of the two families $(Q_0^{a,y}, y > 0)$ and $(Q_0^{(y)}, y > 0)$:*

$$Q_0^{a,\varphi}(\cdot) = \frac{1}{\int_{a_+}^{\infty}(2y-a)\varphi(y)dy} \int_{a_+}^{\infty} (2y-a)\varphi(y) Q_0^{a,y}(\cdot)dy \tag{1.18}$$

$$= \frac{1}{\int_{a_+}^{\infty}(2y-a)\varphi(y)dy} \Big[\int_{a_+}^{\infty} (y-a)\varphi(y) Q_0^{(y)}(\cdot)dy + \int_0^{\infty} \big(1 - \Phi\big(z \vee (a_+)\big)\big) Q_0^{(z)}(\cdot)dz \Big]. \tag{1.19}$$

We would like to generalize Theorem 1.5, by replacing the weight-process $\big(\varphi(S_t)\big)$ with $\big(f(X_t, S_t)\big)$, where $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is Borel.

Theorem 1.6 *To $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that :*

$$\overline{f} := \int_{\mathbb{R}} da \int_{a_+}^{\infty} (2y-a) f(a,y) dy < \infty \tag{1.20}$$

we associate $f^\star = 1/\overline{f}$, and :

$$\varphi(y) = f^\star \Big[\int_{\mathbb{R}} da \int_{y \vee a_+}^{\infty} f(a,\eta) d\eta + \int_{-\infty}^{y} f(a,y)(y-a) da \Big]. \tag{1.21}$$

1. *For every $u \geq 0$, and Γ_u in $\mathcal{F}_u$,*

$$\lim_{t \to \infty} \frac{E_0\big[1_{\Gamma_u} f(X_t, S_t)\big]}{E_0\big[f(X_t, S_t)\big]} = Q_0^{\varphi}(\Gamma_u), \tag{1.22}$$

where Q_0^{φ} is the p.m. introduced in Theorem 1.1, associated with the $\big(P_0, (\mathcal{F}_t)\big)$ martingale (M_t^{φ}).

2. *Moreover the following relations hold :*

$$M_t^{\varphi} = f^\star \int_{\mathbb{R}} da \int_{a_+}^{\infty} (2y-a) f\big(a + X_t, S_t \vee (y + X_t)\big) dy, \tag{1.23}$$

$$Q_0^{\varphi}(\cdot) = f^\star \int_{\mathbb{R}} da \int_{a_+}^{\infty} (2y-a) f(a,y) Q_0^{a,y}(\cdot) dy, \tag{1.24}$$

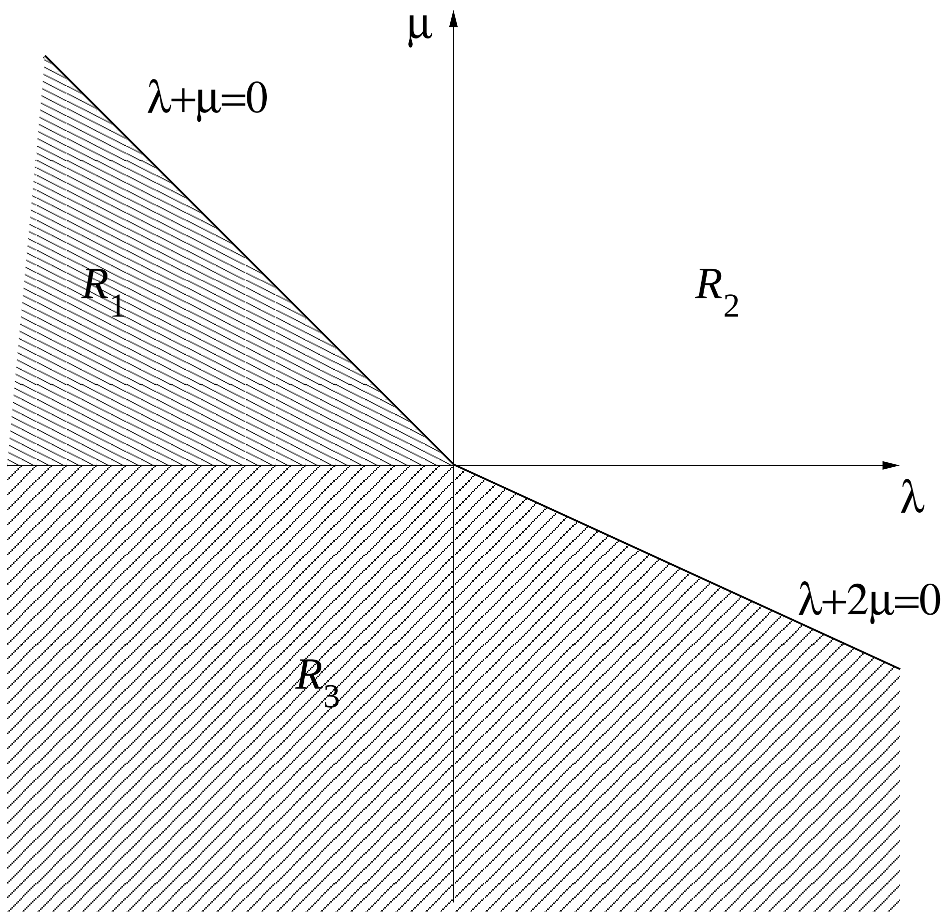
where the p.m. $Q_0^{a,y}$ is defined in Theorem 1.3.

Theorem 1.6 led us to go further and to enquire what happens if $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ does not satisfy (1.20). Rather than trying to give a complete answer, we shall restrict ourselves to functions f of exponential type :

$$f(a,y) = e^{\lambda y + \mu a}, \ y \geq a_+, \lambda, \mu \in \mathbb{R}. \tag{1.25}$$

It is easy to check (see Section 5) that, if f is given by (1.25), then : $\overline{f} < \infty$ iff $\mu > 0$ and $\lambda + \mu < 0$. Then in this case Theorem 1.6 applies.
We claim that for any $\lambda, \mu \in \mathbb{R}$ a penalization principle holds and we are able to describe the limiting p.m. Before stating this result in Theorem 1.7 below, let us introduce the three disjoint sets :

$$R_1 = \big\{(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}; \ \lambda + \mu < 0, \mu \geq 0\big\}, \tag{1.26}$$



$$R_2 = \big\{(\lambda,\mu) \in \mathbb{R}\times\mathbb{R};\ \lambda+2\mu \geq 0, \lambda+\mu \geq 0\big\}, \tag{1.27}$$

$$R_3 = \big\{(\lambda,\mu) \in \mathbb{R}\times\mathbb{R};\ \lambda+2\mu < 0, \mu < 0\big\}. \tag{1.28}$$

See the figure below.

Theorem 1.7 *Let $\lambda, \mu \in \mathbb{R}$.*

1. *For every $u \geq 0$, and Γ_u in $\mathcal{F}_u$,*

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Gamma_u} e^{\mu X_t + \lambda S_t})\big]}{E_0\big[e^{\mu X_t+\lambda S_t}\big]}, \tag{1.29}$$

exists and is equal to $E_0\big[1_{\Gamma_u} M_u^{\mu,\lambda}\big]$, with $(M_u^{\mu,\lambda})$ a positive $\big((\mathcal{F}_u), P_0\big)$ martingale, such that $M_0^{\mu,\lambda} = 1$, which is given by

$$M_u^{\mu,\lambda} = \begin{cases} -(\lambda+\mu)e^{(\lambda+\mu)S_u}(S_u - X_u) + e^{(\lambda+\mu)S_u} & \text{if } (\lambda,\mu) \in R_1, \\ e^{\{(\lambda+\mu)X_u - (\lambda+\mu)^2 u/2\}} & \text{if } (\lambda,\mu) \in R_2, \\ e^{\{(\lambda+\mu)S_u - \mu^2 u/2\}}\Big[\cosh\big(\mu(S_u - X_u)\big) - \dfrac{\lambda+\mu}{\mu}\sinh\big(\mu(S_u-X_u)\big)\Big] & \text{if } (\lambda,\mu) \in R_3. \end{cases}$$

(1.30)

2. *Consequently,* $\Gamma_u(\in \mathcal{F}_u) \mapsto E_0\big[1_{\Gamma_u} M_u^{\mu,\lambda}\big]$ *induces a p.m. on* $\big(\Omega, \mathcal{F}_\infty\big)$.

Remark 1.8 1. *We have already observed that if* f *is defined by (1.25), then* $\overline{f} < \infty$ *iff* $\mu > 0$ *and* $\lambda + \mu < 0$. *Thus, in this case, Theorem 1.6 implies that* $(M_t^{\mu,\lambda})$ *is a martingale of the type* (M_t^φ) *where* φ *is given by (1.21). An easy calculation yields :* $\varphi(y) = -(\lambda+\mu)e^{(\lambda+\mu)y}, y \geq 0$, *and :*

$$M_t^{\mu,\lambda} = M_t^\varphi = -(\lambda+\mu)e^{(\lambda+\mu)S_t}(S_t - X_t) + e^{(\lambda+\mu)S_t},\ t \geq 0.$$

2. *In the third case (i.e.* $(\lambda,\mu) \in R_3$*), the martingale belongs to the family of Kennedy martingales. These martingales were used in [1] and play a central role in [15]. Let us briefly recall the definition of these processes.*

   To $\psi : \mathbb{R} \mapsto [0,\infty[$, *a Borel function satisfying :*

$$\int_x^\infty \psi(z)e^{-\lambda z}dz < \infty, \quad \forall x \in \mathbb{R}. \tag{1.31}$$

   we associate the function $\Phi : \mathbb{R} \mapsto \mathbb{R}$ *:*

$$\Phi(y) = 1 - e^{\lambda y}\int_y^\infty \psi(z)e^{-\lambda z}dz,\ y \in \mathbb{R}. \tag{1.32}$$

   Let φ *be the derivative of* Φ*; then,* $\varphi(y) := \Phi'(y) = \psi(y) - \lambda e^{\lambda y}\int_y^\infty \psi(z)e^{-\lambda z}dz$, *and*

$$M_t^{\lambda,\varphi} := \Big\{\psi(S_t)\frac{\sinh\big(\lambda(S_t - X_t)\big)}{\lambda} + e^{\lambda X_t}\int_{S_t}^\infty \psi(z)e^{-\lambda z}dz\Big\}e^{-\lambda^2 t/2}, \tag{1.33}$$

   is a positive $\big((\mathcal{F}_t), P_0\big)$*-martingale.*

3. *Let* $Q_0^{\mu,\lambda}$ *be the p.m. defined in point 2. of Theorem 1.7, and* P_0^δ *be the law of Brownian motion with drift* δ*, starting at* 0*. Using Theorem 3.9 of [15], we may reformulate (1.30) as follows :*

$$Q_0^{\mu,\lambda} = \begin{cases} Q_0^{\varphi_{-(\mu+\lambda)}} & \text{if } (\lambda,\mu) \in R_1, \\ P_0^{\mu+\lambda} & \text{if } (\lambda,\mu) \in R_2, \\ \dfrac{\lambda+2\mu}{2\mu}e^{\lambda S_\infty} \cdot P_0^\mu & \text{if } (\lambda,\mu) \in R_3 \end{cases} \tag{1.34}$$

   where $\varphi_\delta(y) = \delta e^{-\delta y},\ \delta > 0, y \geq 0$.

The proof of Theorem 1.7 is postponed to Section 5.

Let φ as in Theorem 1.1. We are now interested in the rate of convergence of $Q_{0,t}^\varphi(\Gamma_u) := \dfrac{E_0[1_{\Gamma_u}\varphi(S_t)]}{E_0[\varphi(S_t)]}$ towards $Q_0^\varphi(\Gamma_u)$, as $t \to \infty$, for any $\Gamma_u \in \mathcal{F}_u$. More generally, under additional assumptions, we are able to determine the asymptotic development of $Q_{0,t}^\varphi(\Gamma_u)$ in powers of $1/t$, $t \to \infty$.

Theorem 1.9 *Let* $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *satisfying (1.3) and the related function* Φ *as in Theorem 1.1. We suppose that there exists an integer* $n \geq 1$ *such that :*

$$\int_0^\infty y^{2n+3}\varphi(y)dy < \infty. \tag{1.35}$$

1. *There exists a family of functions $(F_i^\varphi)_{1\le i\le n}$, $F_i^\varphi : \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \mapsto \mathbb{R}$, such that*
   (a) *$(F_i^\varphi(X_t,S_t,t), t\ge 0)$ is a $\big((\mathcal{F}_t),P_0\big)$-martingale, for any $1\le i\le n$,*
   (b) *If $i=1$, we have :*

$$F_1^\varphi(X_t,S_t,t) = -\widetilde{F}_1^\varphi(X_t,S_t) + \big(t+\int_0^\infty y^2\varphi(y)dy\big)M_t^\varphi, \tag{1.36}$$

where

$$\widetilde{F}_1^\varphi(a,y) = \varphi(y)\frac{(y-a)^3}{3!} + \frac{1}{2}\int_y^\infty \varphi(v)(v-a)^3dv, \quad t,y\ge 0, x\in\mathbb{R}. \tag{1.37}$$

2. *The following asymptotic development holds :*

$$\frac{E_0[1_{\Gamma_u}\varphi(S_t)]}{E_0[\varphi(S_t)]} = Q_0^\varphi(\Gamma_u) + \sum_{i=1}^n \frac{1}{t^i}E_0\big[1_{\Gamma_u}F_i^\varphi(X_u,S_u,u)\big] + O\big(\frac{1}{t^{n+1}}\big), \quad t\to\infty. \tag{1.38}$$

Theorem 1.9 will be proved in Section 6. We also give a complement of Theorem 1.9 (Theorem 6.3 in Section 6), taking as weight-process : $\psi(S_t)e^{\lambda(S_t-X_t)}$, with $\lambda>0$.

2 Proof of Theorem 1.2, and of Theorem 1.1, as a consequence

Our proof of Theorem 1.2 is based on the following Lemma.

Lemma 2.1 *Let $y>0$, $u\ge 0$, $\Gamma_u\in\mathcal{F}_u$ and $t>u$. Then :*

$$P_0(\Gamma_u|S_t=y) = \frac{p_{S_u}(y)}{p_{S_t}(y)}E_0\Big[1_{\Gamma_u}h(t-u,y-X_u)|S_u=y\Big]$$

$$+\frac{1}{p_{S_t}(y)}E_0\Big[1_{\Gamma_u}1_{\{S_u<y\}}p_{S_{t-u}}(y-X_u)\Big]. \tag{2.1}$$

where p_{S_r} denotes the density function of S_r, for a fixed $r>0$:

$$p_{S_r}(z) = \sqrt{\frac{2}{\pi r}}e^{-z^2/2r}1_{\{z>0\}}, \tag{2.2}$$

and

$$h(r,z) = P(S_r<z) = \int_0^z p_{S_r}(x)dx = \sqrt{\frac{2}{\pi r}}\int_0^z e^{-x^2/2r}dx, \quad r,z>0. \tag{2.3}$$

Proof of Lemma 2.1 Let $u\ge 0$, $\Gamma_u\in\mathcal{F}_u$ and $t>u$. It is clear that :

$$S_t = S_u \vee \big(X_u + \max_{0\le v\le t-u}\{X_{u+v}-X_u\}\big). \tag{2.4}$$

Consequently if $g:[0,+\infty[\to[0,+\infty]$ is Borel, applying the Markov property at time u leads to :

$$E_0\big[1_{\Gamma_u}g(S_t)\big] = E_0\big[1_{\Gamma_u}\tilde{g}(X_u,S_u)\big],$$

where

$$\tilde{g}(x,y) = E_0\big[g(y\vee\{x+S_{t-u}\})\big], \quad x_+\le y.$$

Then we easily obtain :

$$\begin{aligned}\tilde{g}(x,y) &= g(y)P_0(S_{t-u}\le y-x) + E_0\big[g(x+S_{t-u})1_{\{S_{t-u}>y-x\}}\big]\\ &= g(y)h(t-u,y-x) + \int_0^\infty g(z)p_{S_{t-u}}(z-x)1_{\{z>y\}}dz.\end{aligned}$$

This proves (2.1). ■

Proof of Theorem 1.2 The two estimates :

$$p_{S_t}(y) \sim \sqrt{\frac{2}{\pi t}}, \quad h(t,y) \sim y\sqrt{\frac{2}{\pi t}}, \quad t \to \infty \ (y > 0), \tag{2.5}$$

directly imply that $Q_{0,t}^{(y)}$ converges weakly to $\widetilde{Q}_0^{(y)}$, as $t \to \infty$, where :

$$\widetilde{Q}_0^{(y)}(\Gamma_u) = p_{S_u}(y) E_0\Big[1_{\Gamma_u}(y - X_u) | S_u = y\Big] + E_0[1_{\Gamma_u} 1_{\{S_u < y\}}], \quad \forall u \geq 0 \text{ and } \Gamma_u \in \mathcal{F}_u.$$

Thanks to (1.3), $1 - \Phi(y) = \displaystyle\int_y^\infty \varphi(z)dz, y \geq 0$, then :

$$\begin{aligned}
\int_0^\infty \widetilde{Q}_0^{(y)}(\Gamma_u)\varphi(y)dy &= E_0\Big[1_{\Gamma_u}\big((S_u - X_u)\varphi(S_u) + 1 - \Phi(S_u)\big)\Big] \\
&= E_0\big[1_{\Gamma_u} M_u^\varphi\big] = Q_0^\varphi(\Gamma_u).
\end{aligned}$$

Consequently (1.7) implies $\widetilde{Q}_0^{(y)} = Q_0^{(y)}$, $Q_0^\varphi(S_\infty \in dy)$ a.e.

■

Remark 2.2 *It is interesting to point out that (1.9) permits to prove that $y \mapsto Q_0^{(y)}$ is continuous, as the space of p.m.'s on the canonical space is endowed with the topology of weak convergence.*

As indicated in Section 1, we now show how to prove Theorem 1.1, i.e. how to recover (1.4) from Theorem 1.2 and (1.7).
Indeed, let φ be as in Theorem 1.1. We have :

$$\frac{E_0\big[1_{\Gamma_u}\varphi(S_t)\big]}{E_0\big[\varphi(S_t)\big]} = \frac{\int_0^\infty Q_{0,t}^{(y)}(\Gamma_u)\varphi(y)p_{S_t}(y)dy}{\int_0^\infty \varphi(y)p_{S_t}(y)dy},$$

where $u \geq 0, \Gamma_u \in \mathcal{F}_u$ and $t > u$.
Using Theorem 1.2, (2.5) and the dominated convergence theorem, we get :

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Gamma_u}\varphi(S_t)\big]}{E_0\big[\varphi(S_t)\big]} = \frac{\int_0^\infty Q_0^{(y)}(\Gamma_u)\varphi(y)dy}{\int_0^\infty \varphi(y)dy} = \int_0^\infty Q_0^{(y)}(\Gamma_u)\varphi(y)dy.$$

3 Penalization for long Brownian bridges perturbed by their one-sided maximum

We keep the notation given in Sections 1 and 2.
Let $Q_{0,t}^x$ be the law of the Brownian bridge started at 0, ending at x, with length t :

$$Q_{0,t}^x(\Gamma_t) := E_0[\Gamma_t | X_t = x], \quad \Gamma_t \in \mathcal{F}_t. \tag{3.1}$$

(note the difference with the p.m. $Q_{0,t}^{(x)}$ defined in (1.8)).
Here, we make a simple remark concerning the weak limit of $Q_{0,t}^x$ as $t \to \infty$.
Indeed, we observe that this limit is equal to the Wiener measure P_0 : if $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$ then :

$$\lim_{t\to\infty} Q_{0,t}^x(\Gamma_u) = P_0(\Gamma_u), \tag{3.2}$$

which follows from the fact that $(X_s, 0 \leq s \leq u)$ under $Q_{0,t}^x$, may be represented as $(B_s - \dfrac{s}{t}B_t + \dfrac{s}{t}x, , 0 \leq s \leq u)$, where (B_s) is a Brownian motion started at 0.
The asymptotic study of long Brownian bridges penalized by their one-sided maximum is more involved; in fact, we determine the weak limit $Q_0^{a,y}$ of $Q_{0,t}^{a,y}$ as $t \to \infty$, where $Q_{0,t}^{a,y}$ is the p.m. defined in (1.11). The result is stated in Theorem 1.3.
We proceed as for the proof of Theorem 1.2. We need to generalize Lemma 2.1, taking conditional expectations with respect to (S_t, X_t).

Lemma 3.1 *Let $a \in \mathbb{R}, y > a_+,\ u \geq 0,\ \Gamma_u \in \mathcal{F}_u$ and $t > u$. Then :*

$$P_0(\Gamma_u | X_t = a, S_t = y) = \frac{p_{S_u}(y)}{p_{X_t,S_t}(a,y)} E_0\Big[1_{\Gamma_u}\Big(\int_{\mathbb{R}_+} p_{X_{t-u},S_{t-u}}(a - X_u, \xi) 1_{\{\xi < y - X_u\}} d\xi\Big) \big| S_u = y\Big]$$

$$+ \frac{1}{p_{X_t,S_t}(a,y)} E_0\Big[1_{\Gamma_u} 1_{\{S_u < y\}} p_{X_{t-u},S_{t-u}}(a - X_u, y - X_u)\Big]. \tag{3.3}$$

where p_{X_v,S_v} denotes the density function of $(X_v, S_v), v > 0$:

$$p_{X_v,S_v}(a,y) = \sqrt{\frac{2}{\pi v^3}}(2y - a)e^{-(2y-a)^2/2v} 1_{\{y > a_+\}}, \tag{3.4}$$

Proof. We imitate the proof of Lemma 2.1.
Let $g : [0, +\infty[\times[0, +\infty[\to [0, +\infty]$ be a Borel function. Thanks to (2.4), we have :

$$E_0\big[1_{\Gamma_u} g(X_t, S_t)\big] = E_0\big[1_{\Gamma_u} \tilde{g}(X_u, S_u)\big],$$

where

$$\tilde{g}(a,y) = E_0\big[g(a + X_{t-u}, y \vee \{a + S_{t-u}\})\big], \quad a_+ \leq y.$$

It follows :

$$\tilde{g}(a,y) = \tilde{g}_1(a,y) + \tilde{g}_2(a,y),$$

with :

$$\tilde{g}_1(a,y) = E_0\big[g(a + X_{t-u}, y) 1_{\{S_{t-u} \leq y - a\}}\big],$$
$$\tilde{g}_2(a,y) = E_0\big[g(a + X_{t-u}, a + S_{t-u}) 1_{\{S_{t-u} > y - a\}}\big].$$

Since :

$$\tilde{g}_1(a,y) = \int_{\mathbb{R}\times\mathbb{R}_+} g(b,y) p_{X_{t-u},S_{t-u}}(b - a, \xi) 1_{\{\xi < y - a\}} db d\xi,$$

and

$$\tilde{g}_2(a,y) = \int_{\mathbb{R}\times\mathbb{R}_+} g(b,z) p_{X_{t-u},S_{t-u}}(b - a, z - a) 1_{\{z > y\}} db dz,$$

then (3.3) follows immediately.

■

Proof of Theorem 1.3 Let $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$.
1) Using (2.5) and

$$p_{X_t,S_t}(a,y) \sim \sqrt{\frac{2}{\pi t^3}}(2y - a),\ t \to \infty, \quad y \geq a_+, \tag{3.5}$$

we get :

$$\lim_{t\to\infty} Q_{0,t}^{a,y}(\Gamma_u) = \frac{p_{S_u}(y)}{2y - a} E_0\Big[1_{\Gamma_u}\Big(\int_{(a-X_u)_+}^{y-X_u} (2\xi - a + X_u) d\xi\Big) \big| S_u = y\Big] + \frac{1}{2y - a} E_0\Big[1_{\Gamma_u} 1_{\{S_u < y\}} \big(2y - a - X_u\big)\Big].$$

The first integral in the right-hand side of the previous identity may be computed, which yields :

$$Q_0^{a,y}(\Gamma_u) = p_{S_u}(y) \frac{y - a}{2y - a} E_0\big[1_{\Gamma_u}(y - X_u) | S_u = y\big] + \frac{1}{2y - a} E_0\Big[1_{\Gamma_u} 1_{\{S_u < y\}} \big(2y - a - X_u\big)\Big].$$

2) The relations (1.9) and (2.2) imply :

$$\begin{aligned}(2y-a)Q_0^{a,y}(\Gamma_u) &= (y-a)\{Q_0^{(y)}(\Gamma_u) - E_0\Big[1_{\Gamma_u}1_{\{S_u<y\}}\Big]\} + E_0\Big[1_{\Gamma_u}1_{\{S_u<y\}}\big(2y-a-X_u\big)\Big] \\ &= (y-a)Q_0^{(y)}(\Gamma_u) + E_0\Big[1_{\Gamma_u}1_{\{S_u<y\}}\big(y-X_u\big)\Big].\end{aligned}$$

Applying (1.7) with $\varphi_y = \dfrac{1}{y}1_{[0,y]}$, we get :

$$\int_0^y Q_0^{(z)}(\Gamma_u)dz = yE_0\Big[1_{\Gamma_u}M_u^{\varphi_y}\Big].$$

But $\Phi_y(z) := \displaystyle\int_0^z \varphi_y(r)dr = \frac{z\wedge y}{y}$, consequently :

$$M_u^{\varphi_y} = (S_u - X_u)\varphi_y(S_u) + 1 - \Phi_y(S_u) = \frac{y-X_u}{y}1_{\{S_u<y\}}.$$

This proves (1.13).

■

We now consider the Brownian bridge penalized by a function of its one-sided maximum (cf Theorem 1.5).

Proof of Theorem 1.5
Theorem 1.5 is a direct consequence of Theorem 1.3.
Let $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$ and φ as in Theorem 1.5.
The relations (2.2) and (3.4) imply :

$$P(S_t \in dy|X_t = a) = \frac{2}{t}(2y-a)e^{-\frac{2y(y-a)}{t}}1_{\{y>a_+\}}dy.$$

Consequently :

$$E_0\big[1_{\Gamma_u}\varphi(S_t)|X_t = a\big] = \frac{2}{t}\int_0^{a_+} P_0(\Gamma_u|X_t = a, S_t = y)\varphi(y)(2y-a)e^{-\frac{2y(y-a)}{t}}dy.$$

Hence :

$$\frac{E_0\big[1_{\Gamma_u}\varphi(S_t)|X_t = a\big]}{E_0\big[\varphi(S_t)|X_t = a\big]} = \frac{\int_0^{a_+} P_0(\Gamma_u|X_t = a, S_t = y)\varphi(y)(2y-a)e^{-\frac{2y(y-a)}{t}}dy}{\int_0^{a_+}\varphi(y)(2y-a)e^{-\frac{2y(y-a)}{t}}dy}.$$

Applying Theorem (1.3) and the dominated convergence theorem we get :

$$Q_0^{a,\varphi}(\Gamma_u) = \frac{\int_0^{a_+}(2y-a)\varphi(y)Q_0^{a,y}(\Gamma_u)dy}{\int_0^{a_+}(2y-a)\varphi(y)dy}.$$

This proves (1.18). As for (1.19), it is a direct consequence of (1.13).

4 Proof of Theorem 1.6

1) Point 1. of Theorem 1.6 is a direct consequence of Lemma 3.1 and Theorem 1.3.
Taking the conditional expectation with respect to (X_t, S_t), we obtain :

$$\frac{E_0\big[1_{\Gamma_u}f(X_t,S_t)\big]}{E_0\big[f(X_t,S_t)\big]} = \frac{\int_{\mathbb{R}} da \int_{a_+}^{\infty} Q_{0,t}^{a,y}(\Gamma_u)p_{X_t,S_t}(a,y)f(a,y)dy}{\int_{\mathbb{R}} da \int_{a_+}^{\infty} p_{X_t,S_t}(a,y)f(a,y)dy}, \tag{4.1}$$

where p_{X_t,S_t} denotes the density function of (X_t, S_t), as given by (3.4).
Since f satisfies (1.20), we may apply the dominated convergence theorem; then taking the limit $t \to \infty$, Theorem 1.3, and (3.5) imply :

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Gamma_u} f(X_t,S_t)\big]}{E_0\big[f(X_t,S_t)\big]} := \widetilde{Q}_0(\Gamma_u),$$

where :

$$\widetilde{Q}_0(\Gamma_u) = f^\star \int_{\mathbb{R}} da \int_{a_+}^{\infty} (2y-a) f(a,y) Q_0^{a,y}(\Gamma_u) dy.$$

2) We need to identify $\widetilde{Q}_0(\cdot)$.
Let φ be the function defined by (1.21) and $\Phi(y) = \int_0^y \varphi(z)dz,\ y \geq 0$.
It is clear that $\varphi \geq 0$, then applying Fubini's theorem, we easily obtain :

$$\Phi(y) = f^\star \Big[\int_{\mathbb{R}\times\mathbb{R}_+} f(a,\eta) 1_{\{\eta > y \vee a_+\}} \big(\eta \wedge y + (\eta - a) 1_{\{\eta < y\}}\big) da d\eta \Big]. \tag{4.2}$$

In particular, taking the limit $y \to \infty$, we get : $\lim_{y\to\infty} \Phi(y) = 1$. This means that φ satisfies (1.3).
Moreover :

$$1 - \Phi(y) = f^\star \Big[\int_{\mathbb{R}\times\mathbb{R}_+} f(a,\eta) 1_{\{\eta > y \vee a_+\}} \big(2\eta - a - y)\big) da d\eta \Big]. \tag{4.3}$$

Applying identity (1.13), we get :

$$\begin{aligned}
\widetilde{Q}_0(\cdot) &= f^\star \int_{\mathbb{R}} da \Big(\int_{a_+}^{\infty} \Big\{ (y-a) Q_0^{(y)}(\cdot) + \int_0^y Q_0^{(\eta)}(\cdot) d\eta \Big\} f(a,y) dy \Big) \\
&= f^\star \int_{\mathbb{R}\times\mathbb{R}_+} \Big\{ (\eta - a) f(a,\eta) 1_{\{\eta > a_+\}} + \int_{\eta \vee a_+}^{\infty} f(a,y) dy \Big\} Q_0^{(\eta)}(\cdot) da d\eta \Big\} \\
&= \int_0^{\infty} \varphi(\eta) Q_0^{(\eta)}(\cdot) d\eta.
\end{aligned}$$

Property (1.7) implies : $\widetilde{Q}_0 = Q_0^{\varphi}$.
3) It remains to prove (1.23).
Let $\widetilde{M}_t$ be the process defined as the right-hand side of (1.23) :

$$\widetilde{M}_t = f^\star \int_{\mathbb{R}} db \int_{b_+}^{\infty} (2y - b) f\big(b + X_t, S_t \vee (y + X_t)\big) dy.$$

Setting : $a = b + X_t$ and $\eta = y + X_t$, we obtain :

$$\widetilde{M}_t = f^\star \int_{\mathbb{R}} \widetilde{\psi}_t(a) da,$$

where :

$$\widetilde{\psi}_t(a) = \int_{\mathbb{R}} (2\eta - X_t - a) f(a, S_t \vee \eta) 1_{\{\eta > a, \eta > X_t\}} d\eta.$$

We have :

$$\begin{aligned}
\widetilde{\psi}_t(a) &= 1_{\{S_t > a\}} f(a,S_t) \int_{a \vee X_t}^{S_t} (2\eta - X_t - a) d\eta + \int_{\mathbb{R}} (2\eta - X_t - a) f(a,\eta) 1_{\{\eta > S_t \vee a\}} d\eta \\
&= 1_{\{S_t > a\}} f(a,S_t) \big(S_t - a \vee X_t\big)\big((S_t + a \vee X_t - X_t - a\big) + \int_{\mathbb{R}} (2\eta - X_t - a) f(a,\eta) 1_{\{\eta > S_t \vee a\}} d\eta \\
&= 1_{\{S_t > a\}} f(a,S_t) \big(S_t - X_t\big)\big((S_t - a\big) + \int_{\mathbb{R}} (2\eta - X_t - a) f(a,\eta) 1_{\{\eta > S_t \vee a\}} d\eta.
\end{aligned}$$

Since $M_t^\varphi = (S_t - X_t)\varphi(S_t) + 1 - \Phi(S_t)$, using (1.21) and (4.3), we get :

$$M_t^\varphi = f^\star \int_{\mathbb{R}} \psi_t(a)da,$$

where :

$$\psi_t(a) = (S_t - X_t)\Big[\int_{\mathbb{R}} f(a,\eta)1_{\{\eta > S_t \vee a_+\}}d\eta + f(a,S_t)(S_t - a)1_{\{a<S_t\}}\Big] + \int_{\mathbb{R}} f(a,\eta)(2\eta - a - S_t)1_{\{\eta > S_t\vee a_+\}}d\eta.$$

It is now clear that $\psi_t(a) = \widetilde{\psi}_t(a)$. Consequently $M_t^\varphi = \widetilde{M}_t$.

■

Remark 4.1 *1. Let f be of the type : $f(a,y) = f_1(a)1_{[0,A]}(y)$, where $A > 0$ and $f_1 :]-\infty, A] \mapsto \mathbb{R}_+$ satisfies $\int_{-\infty}^A (1+|a|)f_1(a)da < \infty$. Then it is easy to check that $\bar{f} = A\int_{-\infty}^A (A-a)f_1(a)da < \infty$, and $\varphi(y) = \frac{1}{A}1_{[0,A]}(y)$.*

2. It is possible to recover the identity (1.13) from Theorem 1.6.
Let f as in Theorem 1.6. Using the first part of the proof of Theorem 1.6, (1.7) and (1.21), we have :

$$f^\star \int_{\mathbb{R}} da \int_{a_+}^\infty (2y-a)f(a,y)Q_0^{a,y}(\cdot)dy = Q_0^\varphi(\cdot) = \int_0^\infty \varphi(y)Q_0^{(y)}(\cdot)dy,$$

$$\int_{\mathbb{R}} da \int_{a_+}^\infty (2y-a)f(a,y)Q_0^{a,y}(\cdot)dy = \int_0^\infty Q_0^{(y)}(\cdot)dy\Big\{\int_{\mathbb{R}} da\Big[\int_{y\vee a_+}^\infty f(a,\eta)d\eta + f(a,y)(y-a)1_{\{a<y\}}\Big]\Big\}.$$

Using Fubini's theorem, we easily obtain :

$$\int_{\mathbb{R}} da \int_{a_+}^\infty (2y-a)f(a,y)Q_0^{a,y}(\cdot)dy = \int_{\mathbb{R}} da \int_{a_+}^\infty f(a,y)\Big[y - a + \int_0^y Q_0^{(\eta)}(\cdot)d\eta\Big]dy, \qquad (4.4)$$

for any non-negative function f, satisfying (1.20), but an easy application of Beppo-Levi theorem shows that (4.4) holds even without (1.20) being satisfied.

5 Penalization with $e^{\lambda S_t + \mu X_t}$

1) In this section we focus on penalizations with weight-processes $f(X_t, S_t)$, where the function $f : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ belongs to the family $\{f_{\lambda,\mu};\ f_{\lambda,\mu}(a,y) = e^{\lambda y + \mu a}, \lambda,\mu \in \mathbb{R}\}$.
First, let us determine under which condition $f_{\lambda,\mu}$ satisfies (1.20).
Using the Fubini theorem, we have :

$$\overline{f}_{\lambda,\mu} = \int_0^\infty e^{\lambda y}dy \int_{-\infty}^y (2y-a)e^{\mu a}da.$$

Consequently if $\mu \le 0$ then $\overline{f}_{\lambda,\mu} = \infty$.
Suppose that $\mu > 0$. The integral with respect to da may be computed, this yields to :

$$\overline{f}_{\lambda,\mu} = \frac{1}{\mu^2}\int_0^\infty (1+\mu y)e^{(\lambda+\mu)y}dy.$$

As a result :

$$\overline{f}_{\lambda,\mu} < \infty \quad \text{iff} \quad \mu > 0 \ \text{ and } \ \lambda + \mu < 0. \qquad (5.1)$$

Consequently if this condition holds, then Theorem 1.6 applies.
2) In our approach it is convenient to introduce P_0^μ, the law of Brownian motion with drift μ, starting at 0, and $P_0^{(3)}$ the law of a three dimensional Bessel process started at 0.
Recall Pitman's theorem([8], [4]) :

1. under P_0, the process $\big((2S_t - X_t, S_t),\ t \geq 0\big)$ is distributed as $\big((X_t, J_t),\ t \geq 0\big)$ under $P_0^{(3)}$, where $J_t = \inf_{u \geq t} X_u$.

2. let $(\mathcal{R}_t)$ be the natural filtration associated with the process $\big(R_t = 2S_t - X_t,\ t \geq 0\big)$, then :

$$E_0[f(S_t)|\mathcal{R}_t] = \frac{1}{R_t}\int_0^{R_t} f(u)du, \tag{5.2}$$

for any Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$.

Pitman's theorem has been extended to the case of Brownian motion with drift. From [9] (see also [5]), we know that $(2S_t - X_t, t \geq 0)$ is a diffusion with generator :

$$\frac{1}{2}\frac{d^2}{dx^2} + \mu \coth(\mu x)\frac{d}{dx}. \tag{5.3}$$

Proof of Theorem 1.7

Let u be a fixed positive real number, $\Gamma_u \in \mathcal{F}_u$, and define :

$$\Delta(\Gamma_u, t) := E_0\big[1_{\Gamma_u} e^{\lambda S_t + \mu X_t}\big].$$

1) First suppose that (λ, μ) belongs to R_1. We have already proved that if $\mu > 0$ then Theorem 1.7 is a direct consequence of Theorem 1.6. If $\mu = 0$ and $\lambda + \mu = \lambda < 0$, then Theorem 1.7 follows from Theorem 1.1.

2) We now investigate the last case : $(\lambda, \mu) \in R_3$.

If $\mu = 0$, then $\lambda < 0$ and Theorem 1.7 is a direct consequence of Theorem 1.1.

We suppose, in the sequel $\mu < 0$.

We write $\Delta(\Gamma_u, t)$ as follows :

$$\Delta(\Gamma_u, t) = e^{\mu^2 t/2} E_0^\mu\big[1_{\Gamma_u} e^{\lambda S_t}\big].$$

Applying the Markov property at time u, we get :

$$\Delta(\Gamma_u, t) = e^{\mu^2 t/2} E_0^\mu\big[1_{\Gamma_u} h(X_u, S_u, t-u)\big], \tag{5.4}$$

where

$$h(a, y, r) = E_0^\mu\big[e^{\lambda(y \vee (a + S_r))}\big], \quad y \geq a_+, r \geq 0.$$

Since $\mu < 0$, it is well-known that, under P_0^μ, $X_t \to -\infty$ as $t \to \infty$, $S_\infty < \infty$ and $P_0^\mu(S_\infty > x) = e^{2\mu x}, x \geq 0$.

Consequently :

$$\lim_{r \to \infty} h(a, y, r) = I := -2\mu \int_0^\infty e^{\lambda(y \vee (a+z))} e^{2\mu z} dz.$$

Obviously the above integral may be computed explicitly :

$$\begin{aligned} I &= -2\mu\Big[e^{\lambda y}\int_0^{y-a} e^{2\mu z}dz + e^{\lambda a}\int_{y-a}^\infty e^{(\lambda+2\mu)z}dz\Big] \\ &= e^{\lambda y}\big[1 - e^{2\mu(y-a)} + \frac{2\mu}{\lambda+2\mu}e^{2\mu(y-a)}\big] = e^{\lambda y}\big[1 - \frac{\lambda}{\lambda+2\mu}e^{2\mu(y-a)}\big]. \end{aligned}$$

Moreover, it is easy to check :

$$e^{(\lambda+\mu)y - \mu a}\Big[\cosh\big(\mu(y-a)\big) - \frac{\lambda+\mu}{\mu}\sinh\big(\mu(y-a)\big)\Big] = \frac{\lambda+2\mu}{2\mu}e^{\lambda y}\big[1 - \frac{\lambda}{\lambda+2\mu}e^{2\mu(y-a)}\big]$$

Finally :

$$\lim_{r\to\infty} h(a, y, r) = \frac{2\mu}{\lambda+2\mu}e^{(\lambda+\mu)y-\mu a}\Big[\cosh\big(\mu(y-a)\big) - \frac{\lambda+\mu}{\mu}\sinh\big(\mu(y-a)\big)\Big].$$

Coming back to (5.4), we obtain :

$$\begin{aligned}\lim_{t\to\infty}\frac{E_0\big[1_{\Gamma_u}e^{\mu X_t+\lambda S_t}\big]}{E_0\big[e^{\mu X_t+\lambda S_t}\big]} &= \lim_{t\to\infty}\frac{E_0^\mu\big[1_{\Gamma_u}h(X_u,S_u,t-u)\big]}{E_0^\mu\big[h(0,0,t)\big]}\\ &= E_0^\mu\Big[1_{\Gamma_u}e^{(\lambda+\mu)S_u-\mu X_u}\Big\{\cosh\big(\mu(S_u-X_u)\big)-\frac{\lambda+\mu}{\mu}\sinh\big(\mu(S_u-X_u)\big)\Big\}\Big]\\ &= E_0\big[1_{\Gamma_u}M_u^{\mu,\lambda}\big].\end{aligned}$$

3) Let (λ,μ) be an element of R_2.
a) Let us start with the additional assumption : $\lambda+2\mu>0$. Since

$$P_0^{\lambda+\mu}=e^{(\lambda+\mu)X_t-(\lambda+\mu)^2t/2}P_0\quad\text{on}\quad\mathcal{F}_t,$$

we have :

$$\Delta(\Gamma_u,t)=e^{(\lambda+\mu)^2t/2}E_0^{\lambda+\mu}\big[1_{\Gamma_u}e^{\lambda(S_t-X_t)}\big]. \tag{5.5}$$

Recall Theorem 1.1 in [5] : under $P_0^{\lambda+\mu}$, the process $(S_t-X_t;t\geq 0)$ is distributed as $(|Y_t|,t\geq 0)$, where (Y_t) is the so-called bang-bang process with parameter $\lambda+\mu$, i.e. the diffusion with infinitesimal generator :

$$\frac{1}{2}\frac{d^2}{dx^2}-(\lambda+\mu)sgn(x)\frac{d}{dx}. \tag{5.6}$$

Applying the Markov property at time u in (5.5), yields to :

$$\Delta(\Gamma_u,t)=e^{(\lambda+\mu)^2t/2}E_0^{\lambda+\mu}\big[1_{\Gamma_u}\mathbb{E}_{S_u-X_u}\big\{e^{\lambda|Y_{t-u}|}\big\}\big], \tag{5.7}$$

where $\mathbb{P}_x$ denotes a p.m. under which (Y_t) is the diffusion process with generator (5.6) starting at x. Under $\mathbb{P}_x$, (Y_t) is a recurrent diffusion and $\nu(dx):=(\lambda+\mu)e^{-2(\lambda+\mu)|x|}dx$ is its invariant p.m. Consequently, for any $x\in\mathbb{R}$,

$$\lim_{r\to\infty}\mathbb{E}_x\big[e^{\lambda|Y_r|}\big]=(\lambda+\mu)\int_{\mathbb{R}}e^{\lambda|y|}e^{-2(\lambda+\mu)|y|}dy=\frac{2(\lambda+\mu)}{\lambda+2\mu}.$$

Since $\lambda+2\mu>0$ and $(\lambda,\mu)\in R_2$, then the integral in the right-hand side is finite and does not depend on x. As a result :

$$\lim_{t\to\infty}\frac{E_0\big[1_{\Gamma_u}e^{\mu X_t+\lambda S_t}\big]}{E_0\big[e^{\mu X_t+\lambda S_t}\big]}=\lim_{t\to\infty}\frac{E_0^{\lambda+\mu}\big[1_{\Gamma_u}\mathbb{E}_{S_u-X_u}\big\{e^{\lambda|Y_{t-u}|}\big\}\big]}{\mathbb{E}_0\big[e^{\lambda|Y_t|}\big]}=P_0^{\lambda+\mu}(\Gamma_u).$$

Let us deal with the case $\lambda+2\mu=0,\ \mu\neq 0$. Applying (5.5), we have :

$$\Delta(\Gamma_u,t)=e^{\mu^2t/2}E_0^{-\mu}\big[1_{\Gamma_u}e^{-\mu(2S_t-X_t)}\big].$$

The result follows from Pitman's theorem (for Brownian motion with drift).
b) It remains to study the case : $\lambda+2\mu=0$ and $\lambda>0$.
i) To begin with, we modify $\Delta(\Gamma_u,t)$, λ and μ being for now two real numbers, without restriction. Applying the Markov property at time u leads to :

$$\Delta(\Gamma_u,t)=E_0\big[1_{\Gamma_u}g(X_u,S_u,t-u)\big], \tag{5.8}$$

where

$$g(a,y,r)=E_0\big[e^{\lambda\{y\vee(a+S_r)\}+\mu(a+X_r)}\big],\quad y\geq a_+,r\geq 0.$$

Obviously, $g(a,y,r)$ may be decomposed as follows :

$$g(a,y,r)=e^{\lambda y+\mu a}g_1(a,y,r)+e^{(\lambda+\mu)a}g_2(a,y,r), \tag{5.9}$$

with :

$$g_1(a,y,r) = E_0\big[e^{\mu X_r} 1_{\{S_r<y-a\}}\big], \quad g_2(a,y,r) = E_0\big[e^{\lambda S_r+\mu X_r} 1_{\{S_r\geq y-a\}}\big]. \tag{5.10}$$

Using Pitman's theorem recalled at the beginning of this section, we get :

$$g_1(a,y,r) = E_0^{(3)}\big[e^{-\mu X_r+2\mu J_r} 1_{\{J_r<y-a\}}\big] = E_0^{(3)}\big[e^{-\mu X_r}\frac{1}{X_r}\int_0^{(y-a)\wedge X_r} e^{2\mu z}dz\big], \tag{5.11}$$

$$\begin{aligned} g_2(a,y,r) &= E_0^{(3)}\big[e^{-\mu X_r+(\lambda+2\mu)J_r} 1_{\{J_r\geq y-a\}}\big] \\ &= E_0^{(3)}\big[e^{-\mu X_r} 1_{\{X_r\geq y-a\}}\frac{1}{X_r}\int_{y-a}^{X_r} e^{(\lambda+2\mu)z}dz\big]. \end{aligned} \tag{5.12}$$

As a result, if $\mu\neq 0$:

$$g_1(a,y,r) \sim \frac{e^{2\mu(y-a)}-1}{2\mu}E_0^{(3)}\big[e^{-\mu X_r}\frac{1}{X_r}\big], \quad r\to\infty$$

Recall that :

$$P_0^{(3)}(X_r\in dz) = \sqrt{\frac{2}{\pi r^3}}z^2 e^{-z^2/2r}1_{\{z>0\}}dz. \tag{5.13}$$

Then :

$$E_0^{(3)}\big[e^{-\mu X_r}\frac{1}{X_r}\big] = \sqrt{\frac{2}{\pi r^3}}\int_0^\infty z e^{-\mu z-z^2/2r}dz$$

Setting $b = \dfrac{z+\mu r}{\sqrt{r}}$, we get :

$$E_0^{(3)}\big[e^{-\mu X_r}\frac{1}{X_r}\big] = \sqrt{\frac{2}{\pi r}}e^{\mu^2 r/2}\int_{\mu\sqrt{r}}^\infty (b-\mu\sqrt{r})e^{-b^2/2}db.$$

It turns out that if $\mu<0$:

$$g_1(a,y,r) \sim \big(1-e^{2\mu(y-a)}\big)e^{\mu^2 r/2}, \quad r\to\infty. \tag{5.14}$$

ii) We suppose now that $\lambda=-2\mu>0$.
We need to determine the asymptotic behaviour of $g_2(a,y,r)$ as $r\to\infty$.
Using (5.12) and (5.13) we have :

$$\begin{aligned} g_2(a,y,r) &= E_0^{(3)}\big[e^{-\mu X_r}1_{\{X_r\geq y-a\}}\frac{X_r-y+a}{X_r}\big] \\ &= \sqrt{\frac{2}{\pi r^3}}\int_0^\infty z(z-y+a)e^{-\mu z-z^2/2r}dz \\ &= \sqrt{\frac{2}{\pi r}}e^{\mu^2 r/2}\int_{\mu\sqrt{r}}^\infty (b-\mu\sqrt{r})(\sqrt{r}b-\mu r-y+a)e^{-b^2/2}db. \end{aligned}$$

As a result :

$$g_2(a,y,r) \sim 2\mu^2 r e^{\mu^2 r/2}, \quad r\to\infty. \tag{5.15}$$

Due to (5.8), (5.9), (5.14), (5.15) and $\lambda=-2\mu$, we get :

$$\Delta(\Gamma_u,t) \sim 2\mu^2 t e^{\mu^2 t/2}E_0\big[1_{\Gamma_u}e^{-\mu X_u-\mu^2 u/2}\big], \quad t\to\infty.$$

In particular :

$$\Delta(\Omega,t) = E_0\big[e^{-\mu X_t+\lambda S_t}\big] \sim 2\mu^2 t e^{\mu^2 t/2}, \quad t\to\infty.$$

Finally :

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Gamma_u} e^{\mu X_t+\lambda S_t}\big]}{E_0\big[e^{\mu X_t+\lambda S_t}\big]} = E_0\big[1_{\Gamma_u} e^{-\mu X_u-\mu^2 u/2}\big].$$

■

Remark 5.1 *Here is another proof of Theorem 1.7 : keeping the notations introduced in point 3) b) i) of the proof above, recall that we have proved :*

$$E_0\big[1_{\Gamma_u} e^{\mu X_t+\lambda S_t}\big] = E_0\big[1_{\Gamma_u}\big\{e^{\mu X_u+\lambda S_u} g_1(X_u,S_u,t-u) + e^{(\lambda+\mu)X_u} g_2(X_u,S_u,t-u)\big\}\big],$$

where the functions $g_1(a,y,r)$ and $g_2(a,y,r)$ are given by (5.10) or (5.11) and (5.12). In our proof of Theorem 1.7 we only need the asymptotics of $g_i(a,y,r)$ as $r\to\infty$, $i=1,2$ in the case $\lambda+2\mu=0$, $\lambda>0$. It is actually possible to determine the asymptotics of the previous quantities in any case. However tedious calculations are needed, this explains why we have given a short and direct proof of Theorem 1.7.

We now give a direct interpretation of Theorem 1.7 in terms of the three dimensional Bessel process and its post-minimum.

Proposition 5.2 *Let $\lambda,\mu\in\mathbb{R}$.*

1. *For every $u\geq 0$, and Γ_u in $\mathcal{F}_u$,*

$$\lim_{t\to\infty} \frac{E_0^{(3)}\big[1_{\Gamma_u} e^{\mu X_t+\lambda J_t}\big]}{E_0^{(3)}\big[e^{\mu X_t+\lambda J_t}\big]} := E_0^{(3)}\big[1_{\Gamma_u} \overline{M}_u^{\mu,\lambda}\big], \tag{5.16}$$

where $(\overline{M}_u^{\mu,\lambda})$ is the positive $\big((\mathcal{F}_u),P_0^{(3)}\big)$ martingale :

$$\overline{M}_u^{\mu,\lambda} = \begin{cases} 1 & \text{if } \lambda+\mu<0 \text{ and } \mu\leq 0,\\[2mm] e^{\{-(\lambda+\mu)^2u/2\}}\dfrac{\sinh\big((\lambda+\mu)X_u\big)}{(\lambda+\mu)X_u}\big] & \text{if } \lambda\geq 0 \text{ and } \lambda+\mu\geq 0,\\[2mm] e^{-\mu^2u/2}\dfrac{\sinh\big(\mu X_u\big)}{\mu X_u} & \text{if } \lambda<0 \text{ and } \mu>0, \end{cases} \tag{5.17}$$

Note that $\overline{M}_0^{\mu,\lambda}=1$.

2. *The map : $\Gamma_u(\in\mathcal{F}_u)\mapsto E_0^{(3)}\big[1_{\Gamma_u}\overline{M}_u^{\mu,\lambda}\big]$ induces a p.m. on $\big(\Omega,\mathcal{F}_\infty\big)$.*

Proof. Proposition 5.2 is a direct consequence of Theorem 1.7 and Pitman's theorem.
We have :

$$\frac{E_0^{(3)}\big[1_{\Gamma_u} e^{\mu X_t+\lambda J_t}\big]}{E_0^{(3)}\big[e^{\mu X_t+\lambda J_t}\big]} = \frac{E_0\big[1_{\widehat{\Gamma}_u} e^{-\mu X_t+(\lambda+2\mu)S_t}\big]}{E_0\big[e^{-\mu X_t+(\lambda+2\mu)S_t}\big]}$$

where $\widehat{\Gamma}_u := \{\omega\in\Omega; \widehat{\omega}\in\Gamma_u\}$, and $\widehat{\omega}_t := \sup\limits_{0\leq u\leq t}\omega(u)-\omega(t)$.
Applying our Theorem 1.7, we obtain :

$$\lim_{t\to\infty} \frac{E_0^{(3)}\big[1_{\Gamma_u} e^{\mu X_t+\lambda J_t}\big]}{E_0^{(3)}\big[e^{\mu X_t+\lambda J_t}\big]} = E_0\big[1_{\widehat{\Gamma}_u} M_u^{-\mu,\lambda+2\mu}\big].$$

Since $\widehat{\Gamma}_u \in \mathcal{R}_u$ then :

$$E_0\big[1_{\widehat{\Gamma}_u} M_u^{-\mu,\lambda+2\mu}\big] = E_0\Big[1_{\widehat{\Gamma}_u} E_0\big[M_u^{-\mu,\lambda+2\mu}|\mathcal{R}_u\big]\Big].$$

We claim that :

$$E_0\big[M_u^{-\mu,\lambda+2\mu}|\mathcal{R}_u\big] = \begin{cases} 1 & \text{if } \lambda+\mu<0 \text{ and } \mu\le 0, \\ e^{\{-(\lambda+\mu)^2u/2\}}\dfrac{\sinh\big((\lambda+\mu)(2S_u-X_u)\big)}{(\lambda+\mu)(2S_u-X_u)} & \text{if } \lambda\ge 0 \text{ and } \lambda+\mu\ge 0, \\ e^{-\mu^2u/2}\dfrac{\sinh\big(\mu(2S_u-X_u)\big)}{\mu(2S_u-X_u)} & \text{if } \lambda<0 \text{ and } \mu>0, \end{cases} \tag{5.18}$$

Making again use of Pitman's theorem, it is immediate to obtain (5.17).
As for (5.18), we only prove the third case. The two other cases may be proved similarly. Note that $(\lambda+2\mu,-\mu)\in R_1$ (resp. R_2) iff $\lambda+\mu<0$ and $\mu\le 0$ (resp. $\lambda\ge 0$ and $\lambda+\mu\ge 0$).
As for the third case, we have : $(\lambda+2\mu,-\mu)\in R_3$ iff $\lambda<0$ and $\mu<0$.
Setting $R_u := 2S_u - X_u$, then (1.30)and (5.2) imply :

$$M_u^{-\mu,\lambda+2\mu} = e^{\{(\lambda+\mu)S_u-\mu^2u/2\}}\Big[\cosh\big(\mu(R_u-S_u)\big) - \frac{\lambda+\mu}{\mu}\sinh\big(\mu(R_u-S_u)\big)\Big],$$

$$\begin{aligned} E_0\big[M_u^{-\mu,\lambda+2\mu}|\mathcal{R}_u\big] &= \frac{e^{-\mu^2u/2}}{R_u}\int_0^{R_u} e^{(\lambda+\mu)y}\Big[\cosh\big(\mu(R_u-y)\big) - \frac{\lambda+\mu}{\mu}\sinh\big(\mu(R_u-y)\big)\Big]dy \\ &= \frac{e^{-\mu^2u/2}}{R_u}\Big[-\frac{1}{\mu}e^{(\lambda+\mu)y}\sinh\big(\mu(R_u-y)\big)\Big]_{y=0}^{y=R_u} \\ &= e^{-\mu^2u/2}\frac{\sinh\big(\mu R_u\big)}{\mu R_u}. \end{aligned}$$

This establishes the third case in (5.17), using again Pitman's theorem. ■

Remark 5.3 *It seems natural to ask for :*

$$\lim_{t\to\infty}\frac{E_0^{(3)}\big[1_{\Gamma_u}f(X_t,J_t)\big]}{E_0^{(3)}\big[f(X_t,J_t)\big]}, \tag{5.19}$$

for some suitable Borel $f:\mathbb{R}_+\times\mathbb{R}_+\mapsto\mathbb{R}_+$.
Using Pitman's theorem (see 1. in the proof of Proposition 5.2), the above ratio is equal to :

$$\frac{E_0\big[1_{\widehat{\Gamma}_u}f(2S_t-X_t,S_t)\big]}{E_0\big[f(2S_t-X_t,S_t)\big]}.$$

Consequently Theorem 1.6 applies as soon as :

$$\widetilde{f} := \int_{\mathbb{R}} da\int_{a_+}^{\infty}(2y-a)f(2y-a,y)dy = \int_{\mathbb{R}_+\times\mathbb{R}_+} f(b,y)1_{\{b>y\}}dbdy < \infty. \tag{5.20}$$

Suppose that this condition holds. Then

$$\lim_{t\to\infty}\frac{E_0^{(3)}\big[1_{\Gamma_u}f(X_t,J_t)\big]}{E_0^{(3)}\big[f(X_t,J_t)\big]} = E_0\big[1_{\widehat{\Gamma}_u}M_u^{\varphi}\big],$$

with

$$\varphi(y) = f^{\dagger}\Big[\int_{\mathbb{R}_+\times\mathbb{R}_+} f(b,\eta)1_{\{b>\eta>y\}}dbd\eta + \int_y^{\infty} f(b,y)db\Big],$$

and $f^{\dagger} = 1/\widetilde{f}$.
Proceeding as in the proof of Proposition 5.2 we may prove :

$$E_0\big[M_u^{\varphi}|\mathcal{R}_u\big] = 1. \tag{5.21}$$

Finally the limit in (5.19) equals $P_0^{(3)}(\Gamma_u)$. *In other words the penalization with* $f(X_t, J_t)$, f *satisfying (5.20) does not generate a new p.m.*

As an end to this section, we would like to discuss the relationship between Theorem 1.7 and the results obtained in [3]. Recall that these authors have proved that

$$\lim_{t\to\infty} \frac{E_0^{\mu}\big[1_{\Gamma_u} A_t^{\lambda/2}\big]}{E_0^{\mu}\big[A_t^{\lambda/2}\big]}, \tag{5.22}$$

exists where $\lambda, \mu \in \mathbb{R}$, P_0^{μ} denotes the p.m. on canonical space which makes (X_t) a Brownian motion with drift μ, started at 0, and :

$$A_t = \int_0^t e^{2X_s}ds, \quad t \geq 0.$$

As for our Theorem 1.7, it is proved in [3], that a phase transition phenomenon occurs : there exists three disjoint regions in $\mathbb{R}\times\mathbb{R}$ associated with three types of limit distributions in (5.22). It is striking to note that these regions coincide with the domains R_1, R_2 and R_3 introduced in (1.26)-(1.28).
We have actually no proof of this fact. Nevertheless if $\lambda > 0$, we have a heuristic argument :

$$\frac{E_0^{\mu}\big[1_{\Gamma_u} A_t^{\lambda/2}\big]}{E_0^{\mu}\big[A_t^{\lambda/2}\big]} = \frac{E_0\big[1_{\Gamma_u} e^{\mu X_t} A_t^{\lambda/2}\big]}{E_0\big[e^{\mu X_t} A_t^{\lambda/2}\big]}.$$

Roughly speaking, the Laplace theorem tells us that $A_t = \displaystyle\int_0^t e^{2X_s}ds$ has the same behaviour as e^{2S_t}, see more precisely, the limit results in [2] (formulae (61) and (62) p 181) and [7]. Therefore replacing formally A_t by e^{2S_t}, we get :

$$\frac{E_0\big[1_{\Gamma_u} e^{\mu X_t} A_t^{\lambda/2}\big]}{E_0\big[e^{\mu X_t} A_t^{\lambda/2}\big]} \approx \frac{E_0\big[1_{\Gamma_u} e^{\mu X_t + \lambda S_t}\big]}{E_0\big[e^{\mu X_t+\lambda S_t}\big]}, \quad t \to \infty.$$

6 Asymptotic development

We first recall a penalization result obtained in ([15]), choosing as weight-process : $\psi(S_t)e^{\lambda(S_t - X_t)}$, where $\lambda > 0$.
Let us start with some notations. Let $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a Borel function satisfying :

$$\int_0^{\infty} \psi(z)e^{-\lambda z}dz = 1. \tag{6.1}$$

To ψ we associate the functions Φ and φ :

$$\Phi(y) := 1 - e^{\lambda y}\int_y^{\infty} \psi(z)e^{-\lambda z}dz, \quad y \geq 0, \tag{6.2}$$

$$\varphi(y) := \Phi'(y) = \psi(y) - \lambda e^{\lambda y}\int_y^{\infty} \psi(z)e^{-\lambda z}dz \quad y \geq 0. \tag{6.3}$$

Then :

$$M_t^{\lambda,\varphi} := \Big\{\psi(S_t)\frac{\sinh\big(\lambda(S_t - X_t)\big)}{\lambda} + e^{\lambda X_t}\int_{S_t}^{\infty}\psi(z)e^{-\lambda z}dz\Big\}e^{-\lambda^2 t/2} \quad t \geq 0, \tag{6.4}$$

is a $\big((\mathcal{F}_t), P_0\big)$ positive, and continuous martingale.
In this setting we have proved (see Theorem 3.9 in [15]).

Proposition 6.1 *Let $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a Borel function satisfying (6.1). Then :*

$$\lim_{t\to\infty}\frac{E_0\Big[1_{\Gamma_u}\psi(S_t)e^{\lambda(S_t - X_t)}\Big]}{E_0\Big[\psi(S_t)e^{\lambda(S_t - X_t)}\Big]} = E_0\big[1_{\Gamma_u}M_u^{\lambda,\varphi}\Big], \tag{6.5}$$

for any $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$.

Remark 6.2 *1. In [15], we have determined the law of (X_t) under the new p.m. $\Gamma_u(\in \mathcal{F}_u) \mapsto E_0\big[1_{\Gamma_u}M_u^{\lambda,\varphi}\big]$. However, this result is not used in the sequel.*

2. If we take $\lambda = 0$ and $\psi = \varphi$, then (6.3) holds, (6.1) corresponds to (1.3) and $(M_t^{0,\varphi})$ coincides with the martingale (M_t^{φ}) defined by (1.5). With these conventions, Proposition 6.1 is an extension of points 1. and 2. of Theorem 1.1.

The aim of this section is to prove that, under suitable assumptions, we can obtain an asymptotic expansion of $t \mapsto \dfrac{E_0\Big[1_{\Gamma_u}\psi(S_t)e^{\lambda(S_t - X_t)}\Big]}{E_0\Big[\psi(S_t)e^{\lambda(S_t - X_t)}\Big]}$ as $t \to \infty$. Note that Proposition 6.1 gives the first term.

Theorem 6.3 *Let $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (6.1). We suppose that there exists an integer $n \geq 1$ such that :*

$$\int_0^{\infty}\psi(y)(1 + y^n)dy < \infty. \tag{6.6}$$

1. *There exists a family of functions $(F_i^{\lambda,\varphi})_{1\leq i\leq n}$, $F_i^{\lambda,\varphi} : \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \mapsto \mathbb{R}$, such that*

   (a) $(F_i^{\lambda,\varphi}(X_t, S_t, t), t \geq 0)$ is a $\big((\mathcal{F}_t), P_0\big)$-martingale, for any $1 \leq i \leq n$,

   (b) If $i = 1$, we have :

$$F_1^{\lambda,\varphi}(X_t, S_t, t) = \frac{c(\lambda,\varphi)}{\lambda^3\sqrt{2\pi}}\Big(M_t^{\varphi_1} - M_t^{\lambda,\varphi}\Big). \tag{6.7}$$

   where $c(\lambda,\varphi) := \displaystyle\int_0^{\infty}\psi(x)(1 - \lambda x)dx$ and

$$\varphi_1(y) := \frac{1}{c(\lambda,\varphi)}\Big(\psi(y) - \lambda\int_y^{\infty}\psi(x)dx\Big), \quad y \geq 0. \tag{6.8}$$

   (Note that $\displaystyle\int_0^{\infty}\varphi_1(y)dy = 1$).

2. *The following asymptotic development as $t \to \infty$, holds :*

$$\frac{E_0\big[1_{\Gamma_u}\psi(S_t)e^{\lambda(S_t - X_t)}\big]}{E_0\big[\psi(S_t)e^{\lambda(S_t - X_t)}\big]} = E_0\big[1_{\Gamma_u}M_u^{\lambda,\varphi}\big] + \frac{e^{-\lambda^2 t/2}}{\sqrt{t}}\Big(\sum_{i=1}^{n}\frac{1}{t^i}E_0\big[1_{\Gamma_u}F_i^{\lambda,\varphi}(X_u, S_u, u)\big] + O\big(\frac{1}{t^{n+1}}\big)\Big). \tag{6.9}$$

Note that the two asymptotic expansions (1.38) and (6.9) are drastically different, depending on whether $\lambda > 0$ or $\lambda = 0$. We have already observed in Remark 6.2, that taking formally $\lambda = 0$ in (6.5) gives (1.4). In other words the first term in (6.9) (with $\lambda = 0$) coincides with the first term in (1.38). However the expansion is expressed in terms of powers $t^{-(i+1/2)}$ instead of t^{-i}.

Proof of Theorems 1.9 and 6.3

1) Let us start with some common features concerning the two cases $\lambda > 0$ and $\lambda = 0$, i.e. $\lambda \geq 0$. We adopt the convention that $\psi = \varphi$ if $\lambda = 0$.

Let u be a fixed positive real number, $\Gamma_u \in \mathcal{F}_u$, and

$$\Delta(\lambda, \Gamma_u, t) := E_0\big[1_{\Gamma_u}\psi(S_t)e^{\lambda(S_t - X_t)}\big],$$

where $\lambda \geq 0$.

Applying the Markov property at time u, we get :

$$\Delta(\lambda, \Gamma_u, t) = E_0\big[1_{\Gamma_u} g(\lambda, X_u, S_u, t-u)\big], \tag{6.10}$$

where

$$g(\lambda, a, y, r) := E_0\big[\psi\big(y \vee (a + S_r)\big)e^{\lambda(y\vee(a+S_r)-a-X_r)}\big], \quad r \geq 0, y \geq a_+, a \in \mathbb{R}.$$

We have :

$$g(\lambda, a, y, r) = \psi(y)e^{\lambda(y-a)}E_0\big[e^{-\lambda X_r}1_{\{S_r < y-a\}}\big] + E_0\big[\psi(a+S_r)e^{\lambda(S_r - X_r)}1_{\{S_r \geq y-a\}}\big].$$

Using Pitman's theorem, we get :

$$\begin{aligned} g(\lambda, a, y, r) &= \psi(y)e^{\lambda(y-a)}E_0^{(3)}\big[e^{-\lambda(2J_r - X_r)}1_{\{J_r < y-a\}}\big] + E_0^{(3)}\big[\psi(a+J_r)e^{\lambda(X_r - J_r)}1_{\{J_r \geq y-a\}}\big] \\ &= E_0^{(3)}\Big[\frac{e^{\lambda X_r}}{X_r}\int_0^{X_r}\big\{\psi(y)e^{\lambda(y-a)}e^{-2\lambda z}1_{\{z<y-a\}} + \psi(a+z)e^{-\lambda z}1_{\{z \geq y-a\}}\big\}dz\Big] \\ &= \psi(y)e^{\lambda(y-a)}\int_0^{y-a} e^{-2\lambda z}h(\lambda, z, r)dz + \int_{y-a}^{\infty} e^{-\lambda z}\psi(a+z)h(\lambda, z, r)dz, \end{aligned} \tag{6.11}$$

where :

$$h(\lambda, z, r) = E_0^{(3)}\big[\frac{e^{\lambda X_r}}{X_r}1_{\{X_r > z\}}\big].$$

Applying (5.13), we get :

$$h(\lambda, z, r) = \sqrt{\frac{2}{\pi r^3}}e^{\lambda^2 r/2}\int_z^\infty b e^{-(b-\lambda r)^2/2r}db = \sqrt{\frac{2}{\pi r}}e^{\lambda^2 r/2}\int_{\frac{z-\lambda r}{\sqrt{r}}}^\infty (\lambda\sqrt{r} + v)e^{-v^2/2}dv. \tag{6.12}$$

2) Suppose that $\lambda = 0$. Therefore we replace in the sequel ψ by φ.

a) Then :

$$h(0, z, r) = \sqrt{\frac{2}{\pi r^3}}\int_z^\infty b e^{-b^2/2r}db = \sqrt{\frac{2}{\pi r}}e^{-z^2/2r},$$

$$g(0, a, y, r) = \sqrt{\frac{2}{\pi r}}\widehat{g}(0, a, y, r),$$

with :

$$\widehat{g}(0, a, y, r) = \varphi(y)\int_0^{y-a} e^{-z^2/2r}dz + \int_y^\infty e^{-(v-a)^2/2r}\varphi(v)dv.$$

Let us introduce :

$$A_i(a, y) := \varphi(y)\frac{(y-a)^{2i+1}}{2i+1} + \int_y^\infty (v-a)^{2i}\varphi(v)dv,$$

(note that $A_i(0,0) = \int_0^\infty v^{2i}\varphi(v)dv$; in particular $A_0(0,0) = 1$).
Using the series development of $e^{-\theta}$ with $\theta \geq 0$, we get :

$$\widehat{g}(0,a,y,r) = \sum_{i=0}^{n} \frac{(-1)^i}{(2r)^i i!} A_i(a,y) + O\big(\frac{1}{r^{n+1}}\big). \tag{6.13}$$

Moreover $\varepsilon \mapsto \widehat{g}(0,a,y,1/\varepsilon)$ is of class C^∞ on $[0,1/2]$ and :

$$\Big|\frac{\partial^i \widehat{g}(0,a,y,1/\varepsilon)}{\partial \varepsilon^i}\Big| \leq k_i\Big(\varphi(y)(y-a)^{2i+1} + \int_0^\infty (v-a)^{2i}\varphi(v)dv\Big). \tag{6.14}$$

Suppose that $t \to \infty$, then :

$$\begin{aligned}
\frac{g(0,a,y,t-u)}{g(0,0,0,t)} &= \big(1-\frac{u}{t}\big)^{-1/2} \frac{\displaystyle\sum_{i=0}^{n} \frac{(-1)^i}{2^i i!} \frac{1}{t^i(1-u/t)^i} A_i(a,y) + O\big(\frac{1}{t^{n+1}}\big)}{\displaystyle\sum_{i=0}^{n} \frac{(-1)^i}{2^i i!} \frac{1}{t^i} A_i(0,0) + O\big(\frac{1}{t^{n+1}}\big)} \\
&= \sum_{i=0}^{n} \frac{1}{t^i} F_i(a,y,u) + \frac{1}{t^{n+1}} R(0,a,y,u,t),
\end{aligned} \tag{6.15}$$

where $t \mapsto R(0,a,y,u,t)$ is bounded, and $F_i(a,y,u)$ may be written in the following form :

$$F_i(a,y,u) = \sum_{j=0}^{i} \alpha_{i,j}(u) A_j(a,y), \tag{6.16}$$

$\alpha_{i,j}(u)$ being some polynomial function.
b) In particular :

$$F_0(a,y,u) = \frac{A_0(a,y)}{A_0(0,0)} = A_0(a,y) = \varphi(y)(y-a) + \int_y^\infty \varphi(v)dv = \varphi(y)(y-a) + \Phi(y), \tag{6.17}$$

(note that F_0 does not depend on u).
To compute $F_1(a,y,u)$ we need the first order term :

$$\frac{g(0,a,y,t-u)}{g(0,0,0,t)} = \frac{A_0(a,y)\Big(1 - \dfrac{1}{2t}\dfrac{A_1(a,y)}{A_0(a,y)}\Big)\Big(1-\dfrac{u}{t}\Big)^{-1/2}}{A_0(0,0)\Big(1-\dfrac{1}{2t}\dfrac{A_1(0,0)}{A_0(0,0)}\Big)} + O\big(\frac{1}{t^2}\big).$$

Recall that $A_0(0,0) = 1$, consequently :

$$\begin{aligned}
F_1(a,y,u) &= \frac{A_0(a,y)}{2}\Big(A_1(0,0) - \frac{A_1(a,y)}{A_0(a,y)} + u\Big) \\
&= \frac{A_1(0,0)+u}{2} A_0(a,y) - \frac{A_1(a,y)}{2} \\
&= \frac{\int_0^\infty v^2\varphi(v)dv + u}{2} A_0(a,y) - \varphi(y)\frac{(y-a)^3}{3!} - \frac{1}{2}\int_y^\infty (v-a)^2\varphi(v)dv.
\end{aligned}$$

c) We would like to obtain some estimates about the remainder term $R(0,a,y,u,t)$ in (6.15), as a function of (a,y).

Taking $t \geq 2u+2$ and setting $\varepsilon = 1/t$ we have : $\varepsilon \leq 1/2$, $\varepsilon u \leq 1/2$, $\dfrac{1}{t-u} = \dfrac{\varepsilon}{1-u\varepsilon} \in [0,1]$
Let $\widehat{g}_1(0,a,y,\varepsilon) := \widehat{g}(0,a,y,(1-u\varepsilon)/\varepsilon),\ \varepsilon \in]0, 1/(2u+2)]$. Then property (6.14) implies :

$$|\frac{\partial^i \widehat{g}_1(0,a,y,\varepsilon)}{\partial \varepsilon^i}| \leq K_n\Big(\sum_{j=1}^{i+1}\Big\{\varphi(y)(y-a)^{2j+1} + \int_0^\infty (v-a)^{2j}\varphi(v)dv\Big\}\Big), \quad 0 \leq i \leq n+1, \quad (6.18)$$

where, from now on, K_n denotes a generic constant, which only depends on u.
Let us introduce $\widehat{g}_2(\varepsilon) := \widehat{g}(0,0,0,1/\varepsilon),\ \varepsilon \in]0, 1/(2u+2)]$, then :

$$|\frac{\partial^i \widehat{g}_2(\varepsilon)}{\partial \varepsilon^i}| \leq K_n \int_0^\infty v^{2i+2}\varphi(v)dv, \quad 0 \leq i \leq n+1. \quad (6.19)$$

Note that $\varepsilon \leq 1/2$, consequently :

$$\widehat{g}_2(\varepsilon) \geq \int_0^\infty e^{-v^2/4}\varphi(v)dv > 0. \quad (6.20)$$

Finally, taking into account (6.18), (6.19), (6.20) and (1.35) we get :

$$\begin{aligned}
\Big|\frac{\partial^{n+1}}{\partial \varepsilon^{n+1}}\Big(\frac{g(0,a,y,1/\varepsilon - u)}{g(0,0,0,1/\varepsilon)}\Big)\Big| \quad &\leq \quad K_n\Big(\sum_{j=1}^{n+1}\Big\{\varphi(y)(y-a)^{2j+1} + \int_0^\infty (v-a)^{2j}\varphi(v)dv\Big\}\Big) \\
|R(0,a,y,u,t)| \quad &\leq \quad K_n\Big(\sum_{j=1}^{n+1}\Big\{\varphi(y)(y-a)^{2j+1} + \int_0^\infty (v-a)^{2j}\varphi(v)dv\Big\}\Big) \quad (6.21)
\end{aligned}$$

d) Using (6.10) and (6.15), we have :

$$\begin{aligned}
\frac{E_0\big[1_{\Gamma_u}\varphi(S_t)\big]}{E_0[\varphi(S_t)]} \quad &= \quad E_0\Big[1_{\Gamma_u}\frac{g(0,X_u,S_u,t-u)}{g(0,0,0,t)}\Big] \\
&= \quad E_0\Big[1_{\Gamma_u}\Big\{\sum_{i=0}^{n}\frac{1}{t^i}F_i(X_u,S_u,u) + \frac{1}{t^{n+1}}R(0,X_u,S_u,u,t)\Big\}\Big].
\end{aligned}$$

Inequality (6.21) implies (1.38) and point 1. b) of Theorem 1.9.
e) It remains to prove that for any $i \in \{1,\cdots,n\}$, $\big(F_i^\varphi(X_t,S_t,t), t \geq 0\big)$ is a $\big((\mathcal{F}_t), P_0\big)$ martingale.
From (6.16), we deduce that $E\big[|F_i^\varphi(X_t,S_t,t)|\big] < \infty$.
Let $\Gamma_u \in \mathcal{F}_u$ and $u \leq v$. The asymptotic development (1.38) implies that :

$$\lim_{t\to\infty} t\Big\{\frac{E_0[1_{\Gamma_u}\varphi(S_t)]}{E_0[\varphi(S_t)]} - Q_0^\varphi(\Gamma_u)\Big\} = E_0\big[1_{\Gamma_u}F_1^\varphi(X_u,S_u,u)\big].$$

Since $\Gamma_u \in \mathcal{F}_v$ then :

$$E_0\big[1_{\Gamma_u}F_1^\varphi(X_u,S_u,u)\big] = E_0\big[1_{\Gamma_u}F_1^\varphi(X_v,S_v,v)\big].$$

Consequently $\big(F_1^\varphi(X_t,S_t,t), t \geq 0\big)$ is a $\big((\mathcal{F}_t), P_0\big)$ martingale.
Reasoning by induction, we easily prove that $\big((F_i^\varphi(X_t,S_t,t), t \geq 0)\big)$ is a $\big((\mathcal{F}_t), P_0\big)$ martingale, for any $1 \leq i \leq n$.
3) We now suppose $\lambda > 0$. We proceed as previously; the relation (6.12) implies :

$$h(\lambda,z,r) = \sqrt{\frac{2}{\pi}}e^{\lambda^2 r/2}\Big[\lambda\sqrt{2\pi} + \frac{1}{\sqrt{r}}e^{-(\frac{z-\lambda r}{\sqrt{r}})^2/2} - \lambda\Phi_0\big(\frac{z-\lambda r}{\sqrt{r}}\big)\Big],$$

where Φ_0 denotes the function : $\Phi_0(x) := \displaystyle\int_{-\infty}^x e^{-u^2/2}du$.

Suppose $x<0$. Integrating by parts we have :

$$\Phi_0(x)=\int_{-\infty}^x \frac{1}{u}ue^{-u^2/2}du=-\frac{1}{x}e^{-x^2/2}-\int_{-\infty}^x \frac{1}{u^2}e^{-u^2/2}du.$$

Reasoning by induction we can easily prove :

$$\Phi_0(x)=e^{-x^2/2}\sum_{i=0}^n(-1)^{i+1}\frac{a_i}{x^{2i+1}}+(-1)^{n+1}a_{n+1}\int_{-\infty}^x \frac{1}{u^{2n+2}}e^{-u^2/2}du,$$

with $a_0=1$ and

$$a_i=1\times 3\times\cdots\times(2i-1)=\frac{(2i)!}{2^i i!}=E_0[X_1^{2i}],\qquad i\geq 1.$$

This relation implies :

$$\Phi_0(x)=e^{-x^2/2}\Big[\sum_{i=0}^n(-1)^{i+1}\frac{a_i}{x^{2i+1}}+O\big(\frac{1}{x^{2n+3}}\big)\Big],\quad x\to-\infty.$$

Consequently :

$$h(\lambda,z,r)=\sqrt{\frac{2}{\pi}}e^{\lambda^2r/2}\Big[\lambda\sqrt{2\pi}+e^{-(\frac{z-\lambda r}{\sqrt{r}})^2/2}\ h_1(\lambda,z,r)\Big]$$

where :

$$h_1(\lambda,z,r):=-\frac{z}{\lambda r^{3/2}(1-\frac{z}{\lambda r})}+\sum_{i=1}^n(-1)^{i+1}\frac{a_i}{\lambda^{2i}}\Big(\frac{1}{\sqrt{r}(1-\frac{z}{\lambda r})}\Big)^{2i+1}+0\big(\frac{1}{r^{n+3/2}}\big),\quad r\to\infty.$$

Setting $r=t-u$, where $u>0$ is fixed and $t\to\infty$, we get :

$$\begin{aligned}h_1(\lambda,z,t-u)\ &=\ -\frac{z}{\lambda t^{3/2}}\frac{1}{(1-u/t)^{3/2}(1-\frac{z}{\lambda t(1-u/t)})}\\ &+\ \sum_{i=1}^n\frac{(-1)^{i+1}a_i}{\lambda^{2i}}\frac{1}{t^{i+1/2}}\Big(\frac{1}{\sqrt{1-u/t}(1-\frac{z}{\lambda t(1-u/t)})}\Big)^{2i+1}+0\big(\frac{1}{t^{n+3/2}}\big),\quad t\to\infty,\end{aligned}$$

This implies :

$$h_1(\lambda,z,t-u)=\frac{1}{\sqrt{t}}\Big[\sum_{i=1}^n\alpha_i(\lambda,z,u)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big)\Big],\quad t\to\infty,$$

where, for any i, $(z,u)\mapsto\alpha_i(\lambda,z,u)$ is a polynomial function with degree less than n, with respect to z or u. Moreover we have :

$$\alpha_1(\lambda,z,u)=-\frac{z}{\lambda}+\frac{1}{\lambda^2}.$$

If $r=t-u$ we have :

$$e^{-(\frac{z-\lambda r}{\sqrt{r}})^2/2}=e^{\lambda z+\lambda^2u/2}e^{-\lambda^2t/2}e^{\frac{z^2}{2t(1-u/t)}},$$

therefore :

$$h(\lambda,z,t-u)=\sqrt{\frac{2}{\pi}}e^{\lambda^2(t-u)/2}\Big[\lambda\sqrt{2\pi}+e^{\lambda z+\lambda^2u/2}e^{-\lambda^2t/2}\frac{1}{\sqrt{t}}\Big\{\sum_{i=1}^n\beta_i(\lambda,z,u)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big)\Big\}\Big],\quad t\to\infty,$$

where $(z,u)\mapsto\beta_i(\lambda,z,u)$ is a polynomial function with at most degree n with respect to z or u. Note that $\beta_1(\lambda,z,u)=\alpha_1(\lambda,z,u)=-\dfrac{z}{\lambda}+\dfrac{1}{\lambda^2}$

We are able to come back to relation (6.11) :

$$\begin{aligned} g(\lambda,a,y,t-u) &= \sqrt{\frac{2}{\pi}}e^{\lambda^2(t-u)/2}\Big[\lambda\sqrt{2\pi}\Big(\psi(y)\frac{\sinh\big(\lambda(y-a)\big)}{\lambda}+e^{\lambda a}\int_y^\infty e^{-\lambda x}\psi(x)dx\Big) \\ &+ e^{\lambda^2 u/2}e^{-\lambda^2 t/2}\frac{1}{\sqrt{t}}\sum_{i=1}^n \gamma_i(\lambda,a,y,u)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big)\Big], \quad t\to\infty, \end{aligned}$$

where

$$\gamma_i(\lambda,a,y,u)=\psi(y)e^{\lambda(y-a)}\int_0^{y-a}e^{-\lambda z}\beta_i(\lambda,z,u)dz+\int_y^\infty \psi(x)\beta_i(\lambda,x-a,u)dx.$$

Note that :

$$|\gamma_i(\lambda,a,y,u)|\le C(1+u^n)\big(1+|a|^n+\psi(y)e^{\lambda(y-a)}\big), \quad a\in\mathbb{R}, y\ge 0, u\ge 0. \tag{6.22}$$

Introducing :

$$F_0^{\lambda,\varphi}(a,y,u):=e^{-\lambda^2u/2}\Big(\psi(y)\frac{\sinh\big(\lambda(y-a)\big)}{\lambda}+e^{\lambda a}\int_y^\infty e^{-\lambda x}\psi(x)dx\Big), \tag{6.23}$$

We have :

$$g(\lambda,a,y,t-u)=\sqrt{\frac{2}{\pi}}e^{\lambda^2t/2}\Big[\lambda\sqrt{2\pi}F_0^{\lambda,\varphi}(a,y,u)+\frac{e^{-\lambda^2t/2}}{\sqrt{t}}\sum_{i=1}^n\gamma_i(\lambda,a,y,u)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big)\Big], \quad t\to\infty,.$$

Since $F_0^{\lambda,\varphi}(0,0,0)=\int_0^\infty e^{-\lambda x}\psi(x)dx=1$, then

$$\begin{aligned} \frac{g(\lambda,a,y,t-u)}{g(\lambda,0,0,t)} &= \frac{\lambda\sqrt{2\pi}F_0^{\lambda,\varphi}(a,y,u)+\dfrac{e^{-\lambda^2t/2}}{\sqrt{t}}\displaystyle\sum_{i=1}^n\gamma_i(\lambda,a,y,u)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big)}{\lambda\sqrt{2\pi}+\dfrac{e^{-\lambda^2t/2}}{\sqrt{t}}\displaystyle\sum_{i=1}^n\gamma_i(\lambda,0,0,0)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big)} \\ &= F_0^{\lambda,\varphi}(a,y,u)+\frac{e^{-\lambda^2t/2}}{\sqrt{t}}\sum_{i=1}^n F_i^{\lambda,\varphi}(a,y,u)\frac{1}{t^i}+0\big(\frac{1}{t^{n+1}}\big), \quad t\to\infty. \end{aligned}$$

In particular :

$$F_1^{\lambda,\varphi}(a,y,u)=\frac{1}{\lambda\sqrt{2\pi}}\Big(\gamma_1(\lambda,a,y,u)-\gamma_1(\lambda,0,0,0)F_0^{\lambda,\varphi}(a,y,u)\Big).$$

It is easy to compute $\gamma_1(\lambda,a,y,u)$. We have :

$$\begin{aligned} \gamma_1(\lambda,a,y,u) &= \frac{1}{\lambda^2}\Big(\psi(y)e^{\lambda(y-a)}\int_0^{y-a}e^{-\lambda z}(1-\lambda z)dz+\int_y^\infty\psi(x)\big(1-\lambda(x-y)-\lambda(y-a)\big)dx\Big) \\ &= \frac{1}{\lambda^2}\Big(c(\lambda,\varphi)\varphi_1(y)(y-a)+\int_y^\infty\psi(x)\big(1-\lambda(x-y)\big)dx\Big), \end{aligned}$$

with $c(\lambda,\varphi)=\int_0^\infty\psi(x)(1-\lambda x)dx$ and

$$\varphi_1(y)=\frac{1}{c(\lambda,\varphi)}\Big(\psi(y)-\lambda\int_y^\infty\psi(x)dx\Big), \quad y\ge 0.$$

Setting :

$$\Phi_1(y) := \int_0^y \varphi_1(z)dz,$$

we obtain :

$$\begin{aligned}\Phi_1(+\infty) - \Phi_1(y) &= \frac{1}{c(\lambda,\varphi)}\Big(\int_y^\infty \psi(x)dx - \lambda\int_y^\infty \psi(x)(x-y)dx\Big)\\ &= \frac{1}{c(\lambda,\varphi)}\Big(\int_y^\infty \psi(x)\big(1-\lambda(x-y)\big)dx\Big).\end{aligned}$$

In particular :

$$\Phi_1(+\infty) - \Phi_1(0) = \int_0^\infty \varphi_1(z)dz = 1.$$

Moreover :

$$F_1^{\lambda,\varphi}(a,y,u) = \frac{c(\lambda,\varphi)}{\lambda^3\sqrt{2\pi}}\Big(\varphi_1(y)(y-a) + \int_y^\infty \varphi_1(z)dz - F_0^{\lambda,\varphi}(a,y,u)\Big).$$

This proves point 1. (b) of Theorem 6.3.

■

7 Further discussions about Brownian penalizations

As a conclusion to this paper, we would like to mention that we are presently developing some further discussions about Brownian penalizations in three papers in preparation :

- in [13], we study a number of extensions of Pitman's theorem, which are closely related with the penalizations found in the present paper;
- in [12], we extend most of the results found in the present paper when (X_t) is replaced with (R_t), a Bessel process with dimension $d < 2$, the weight process being a function of the local time of (R_t) at level 0;
- in [11], we study penalization results for n-dimensional Brownian motion, when the weight process is $\big(\exp - \int_0^t 1_C(X_s)ds\big)$, where C denotes a cone in $\mathbb{R}^d$ with vertex 0.

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